Pergamon

J. Appl. Maths Mechs, Vol. 59, No. 4, pp. 671–676, 1995

0021–8928/95 \$24.00+0.00

0021–8928(95)00078-X

THE STRESSED STATE OF A LAMINATED ELASTIC COMPOSITE WITH A THIN LINEAR INCLUSION†

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(*Received* 10 *June* 1993)

The plane deformation of a piecewise-uniform body is investigated using the method of homogenization of an elastic medium with a macroperiodic structure [1, 2]. The body consists of a periodic system of joined heterogeneous strips, and there is a thin elastic inclusion of finite length in one of the straight interfaces of the materials. By representing the stresses and displacements in terms of complex potentials [3] and using the conditions for the interaction of a thin elastic inclusion with the matrix [4–6] a system of four integrodifferential equations of the problem is obtained which has a solution which is appropriate for inclusions of any stiffness: from absolutely pliable, which simulates a cut, to absolutely rigid. Unlike existing solutions for defects at the interface of materials [7, 8], the solution obtained does not oscillate in the region of the tip of the inclusion.

1. THE HOMOGENIZED MODEL OF A MICROPERIODIC ELASTIC COMPOSITE

We consider the elastic equilibrium of a composite material consisting of a periodic system of bonded elastic heterogeneous strips 1 and 2 (see Fig. 1), acted upon by an arbitrary external load. Suppose λ_j, μ_j ($j = 1, 2$) are the Lamé constants, l_j ($j = 1, 2$) is the width of strips 1 and 2, respectively, and $\delta = l_1$ and l_2 is the period of the composite, and we introduce a rectangular system of coordinates xy, the x axis of which coincides with one of the straight interfaces of the materials.

By the assumptions of the linear theory of elasticity of bodies with microlocal parameters [1, 2], we can represent the components of the vector of elastic displacements in the form

$$U(x,y)=u(x,y)+h(y)p(x,y),\quad V(x,y)=\upsilon(x,y)+h(y)q(x,y) \tag{1.1}$$

Here u and υ are the macrodisplacements in the directions of the x and y axes, respectively, p and q are microlocal parameters which, when there are no mass forces, satisfy the system of differential equations of the equilibrium of a homogenized microperiodic medium [8]

$$(\bar\lambda+\bar\mu)(u_{,xy}+\upsilon_{,yy})+\bar\mu(\upsilon_{,xx}+\upsilon_{,yy})+[\mu]p_{,x}+([\lambda]+2[\mu])q_{,y}=0 \tag{1.2}$$

$$(\bar\lambda+\bar\mu)(u_{,xx}+\upsilon_{,xy})+\bar\mu(u_{,xx}+u_{,yy})+[\mu]p_{,y}+[\lambda]q_{,x}=0$$

$$(\hat\lambda+2\hat\mu)q+[\lambda](u_{,x}+\upsilon_{,y})+2[\mu]\upsilon_{,y}=0,\quad \hat\mu p+[\mu](u_{,y}+\upsilon_{,x})=0 \tag{1.3}$$

$$(\bar\lambda,\bar\mu)\equiv(\langle\lambda\rangle,\langle\mu\rangle),\quad ([\lambda],[\mu])\equiv(\langle\lambda h_{,y}\rangle,\langle\mu h_{,y}\rangle)$$

$$(\hat\lambda,\hat\mu)\equiv(\langle\lambda(h_{,y})^2\rangle,\ \langle\mu(h_{,y})^2\rangle)$$

and $h(\cdot)$ is a specified piecewise-linear real form function having the properties

$$|h(y)|<\delta,\quad h(y+d)=h(y),\quad \int\limits_{y-\delta/2}^{y+\delta/2}h(t)dt=0,\quad \forall y\in R \tag{1.4}$$

The symbol $\langle f(\cdot)\rangle$ denotes the mean value of the δ-periodic function $f(\cdot)$.

Substituting (1.1) into Hooke's law for each of strips 1 and 2 we obtain the expressions

$$\begin{aligned}\sigma_{yj}&=(\lambda_j+2\mu_j)(\upsilon_{,y}+h_{,j}q+hq_{,y})+\lambda_j(u_{,x}+hp_{,x})\\ \sigma_{xj}&=(\lambda_j+2\mu_j)(u_{,x}+hp_{,x})+\lambda_j(\upsilon_{,y}+h_{,j}q+hq_{,y})\\ \sigma_{xyj}&=\mu_j(\upsilon_{,x}+hq_{,x}+u_{,y}+h_{,j}p+hp_{,y})\end{aligned} \tag{1.5}$$

Here $h_{,j}$ is the derivative of h_j' for y belonging to the jth strip.

Since $|h(y)|<\delta$, $\forall y$, for sufficiently small values of δ we can neglect terms containing h in relations (1.2) and

†*Prikl. Mat. Mekh.* Vol. 59, No. 4, pp. 698–703, 1995.

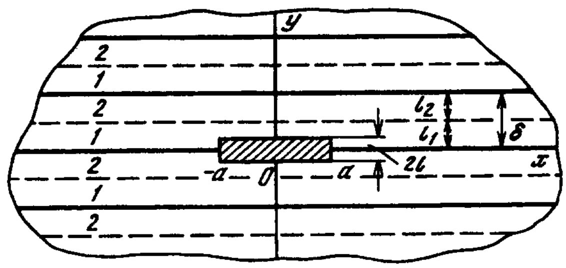

Fig. 1.

(1.5) (a strict proof of this operation using non-standard analysis is given in [2]). Note that we cannot omit terms containing the derivative $h_{,j}$ in (1.5).

Equations (1.1)–(1.5) describe the homogenized model of a microperiodic composite and they depend very much on the form of the form function $h(y)$. For the problem considered the δ-periodic function will be chosen in the form

$$h(y)=\begin{cases} y-l_1/2, & 0\le y\le l_1 \\ [\eta y+(1+\eta)l_1/2]/(1-\eta), & l_1\le y\le\delta \end{cases} \tag{1.6}$$

$$\eta=l_1/\delta\in(0,1/2)$$

Then

$$h_{,j}=\begin{cases} 1, & j=1 \\ -\eta/(1-\eta), & j=2 \end{cases}$$

and it follows from relations (1.3) and (1.4) that

$$\begin{gathered}\bar\lambda=\eta\lambda_1+(1-\eta)\lambda_2,\quad \bar\mu=\eta\mu_1+(1-\eta)\mu_2 \\ [\lambda]=\eta(\lambda_1-\lambda_2),\quad [\mu]=(\mu_1-\mu_2) \\ \hat\lambda=\eta\lambda_1+\eta^2\lambda_2/(1-\eta),\quad \hat\mu=\eta\mu_1+\eta^2\mu_2/(1-\eta)\end{gathered} \tag{1.7}$$

Eliminating the microlocal parameters p and q from (1.2) using (1.3) we obtain the following system of equations for the macrodisplacements u and υ

$$\begin{gathered}A_2u_{,xx}+(B+C)\upsilon_{,xy}+Cu_{,yy}=0,\quad A_1\upsilon_{,yy}+(B+C)u_{,xy}+C\upsilon_{,xx}=0 \\ A_1=\bar\lambda+2\bar\mu-\frac{([\lambda]+2[\mu])^2}{\hat\lambda+2\hat\mu}>0,\quad A_2=\bar\lambda+2\bar\mu-\frac{[\lambda]^2}{\hat\lambda+2\hat\mu}>0 \\ B=\bar\lambda-\frac{[\lambda]([\lambda]+2[\mu])}{\hat\lambda+2\hat\mu}>0,\quad C=\bar\mu-\frac{[\mu]^2}{\hat\mu}>0\end{gathered} \tag{1.8}$$

In a similar way, after eliminating p and q from (1.5) we obtain

$$\begin{gathered}\sigma_{yy}^{(j)}=Bu_{,x}+A_1\upsilon_{,y},\quad \sigma_{xy}^{(j)}=C(u_{,y}+\upsilon_{,x}),\quad \sigma_{xx}^{(j)}=D_j\upsilon_{,y}+E_ju_{,x} \\ D_j=\frac{\lambda_j}{\lambda_j+2\mu_j}A_1,\quad E_j=\frac{4\mu_j(\lambda_j+\mu_j)}{\lambda_j+2\mu_j}+\frac{\lambda_j}{\lambda_j+2\mu_j}B,\quad j=1,2\end{gathered} \tag{1.9}$$

The system of equations (1.8) and relations (1.9) describe the elastic equilibrium of a medium homogenized as given by (1.5)–(1.6). Apart from the coefficients they are identical with the corresponding equations for an anisotropic medium [9]. Hence, the characteristics of the stress–strain state of the composite considered can be expressed, using the well-known procedure in [3, 10], in terms of two holomorphic functions $\Phi_j(\cdot)$ (j = 1, 2). Here we must distinguish two versions of the values of the mechanical constants of the strips 1 and 2:

the case when $\mu_1 \neq \mu_2$. We have (summing from $k = 1$ to $k = 2$)

$$\begin{aligned}
&\sigma_{yj}(x,y) = 2\operatorname{Re}\sum \Phi_k(z_k), \quad \sigma_{xyj}(x,y) = 2\operatorname{Im}\sum t_k\Phi_k(z_k) \\
&\sigma_{xj}(x,y) = 2\operatorname{Re}\sum C_{kj}\Phi_k(z_k), \quad j = 1,2 \\
&u_{,x}(x,y) = -2\operatorname{Re}\sum p_k\Phi_k(z_k), \quad \upsilon_{,x}(x,y) = 2\operatorname{Im}\sum t_k p_{3-k}\Phi_k(z_k) \\
&t_1 = (t_+ - t_-)/2, \quad t_2 = (t_+ + t_-)/2, \quad z_k = x + it_k y, \quad k = 1,2 \\
&t_\pm = \left[\frac{(A_\pm + 2C)A_\pm}{A_1 C}\right]^{1/2}, \quad p_k = \frac{A_1 t_k^2 + B}{A_1 A_2 - B^2} \\
&C_{kj} = \frac{\lambda_j}{\lambda_j + 2\mu_j} - \frac{4\mu_j(\lambda_j + \mu_j)}{\lambda_j + 2\mu_j} p_k, \quad k,j = 1,2; \quad A_\pm = \sqrt{A_1 A_2} \pm B
\end{aligned} \tag{1.10}$$

the case when $\mu_1 = \mu_2$. We have

$$\begin{aligned}
&\sigma_{yj}(x,y) = \operatorname{Re}[\Phi_1(z) + \overline{\Phi_1(z)} + \Phi_2(z)], \quad \sigma_{xyj}(x,y) = \operatorname{Im}[\Phi_2(z)] \\
&\sigma_{xj}(x,y) = \operatorname{Re}[a_j\Phi_1(z) + a_j\overline{\Phi_1(z)} - \Phi_2(z)], \quad j = 1,2 \\
&2C[u_{,x}(x,y) + \upsilon_{,x}(x,y)] = \kappa\Phi_1(z) - \overline{\Phi_1(z)} - \overline{\Phi_2(z)} \\
&\kappa = \frac{A_1 + C}{A_1 - C}, \quad a_j = 1 + \frac{2C(\lambda_j + 2C - A_1)}{(A_1 - C)(\lambda_j + 2C)}, \quad j = 1,2; \quad z = x + iy
\end{aligned} \tag{1.11}$$

2. THE MODEL OF A THIN ELASTIC INCLUSION

The conditions for the interaction of an elastic inclusion of finite length $2a$ and small width $2l$, localized on the straight interface of the materials of strips 1 and 2 (see Fig. 1) have the form [5, 8]

$$\begin{aligned}
&(u_{1,x}^+ + u_{2,x}^-)l = k_0 N - k_1 l(\sigma_{y1}^+ + \sigma_{y2}^-), \quad \upsilon_1^+ - \upsilon_2^- = k_0 l(\sigma_{y1}^+ + \sigma_{y2}^-) - k_1 N \\
&u_1^+ - u_2^- + l(\upsilon_{1,x}^+ + \upsilon_{2,x}^-) = \mu_0^{-1} l(\sigma_{xy1}^+ + \sigma_{xy2}^-) \\
&\sigma_{y1}^+ - \sigma_{y2}^- + Q_{,x} = 0, \quad \sigma_{xy1}^+ - \sigma_{xy2}^- + N_{,x} = 0 \\
&l(\sigma_{xy1}^+ + \sigma_{xy2}^-) - Q(x) = 0 \\
&N(x) = \int_{-l}^{l} \sigma_x(x,y)dy, \quad Q(x) = \int_{-l}^{l} \sigma_{xy}(x,y)dy, \quad k_0 = \frac{1-\nu_0}{2\mu_0}, \quad k_1 = \frac{\nu_0}{2\mu_0}
\end{aligned} \tag{2.1}$$

Here μ_0 is the shear modulus, ν_0 is Poisson's ratio of the material of the inclusion, the superscript plus relates to the corresponding quantities on the upper side of the inclusion and the superscript minus refers to quantities on the lower side of the inclusion, $N(x)$ is the axial force and $q(x)$ is the cutting force in an arbitrary section of the inclusion. The system of equations (2.1) describes the deformation of the surfaces of the inclusion when acted upon by applied external forces and, with accuracy $0(l^2)$, takes into account the deformation of longitudinal tension and transverse shear of the layer [6].

3. THE SYSTEM OF INTEGRAL EQUATIONS

Since the problem in question is linear we have

$$\sigma = \sigma^0 + \sigma^*, \quad u = u^0 + u^* \tag{3.1}$$

(σ^0 and u^0 are the known tensor and vector of the elastic displacements of the multilayer composite without the inclusion and σ^* and u^* is the solution of the corresponding perturbed problem). Since we are dealing with a thin inclusion, its presence can be simulated by sudden changes in the stresses and the derivatives of the displacements on the axial line of the layer

$$\sigma^*_{y1}(x,+0)-\sigma^*_{y2}(x,-0)=f_1^*(x),\quad \sigma^*_{xy1}(x,+0)-\sigma^*_{xy2}(x,-0)=f_2^*(x)$$
$$u^*_{,x}(x,+0)-u^*_{,x}(x,-0)=f_3^*(x),\quad \upsilon^*_{,x}(x,+0)-\upsilon^*_{,x}(x,-0)=f_4^*(x) \tag{3.2}$$
$$f_k^*(x)=\begin{cases} f_k(x), & |x|\le a \\ 0, & |x|>a \end{cases}\quad k=1,2,3,4$$

We obtain the following results from conditions (3.2) using relations (1.10), after solving the corresponding problems of linear coupling for the different versions considered in Section 1.

The case when $\mu_1 \neq \mu_2$. We have

$$\Phi_j(z_j)=\frac{(-1)^j}{p_1-p_2}\left[p_{3-j}w_1(z_j)-i\frac{p_j}{t_j}w_2(z_j)+w_3(z_j)+i\frac{1}{t_j}w_4(z_j)\right]$$
$$w_k(z_j)=\frac{1}{2\pi i}\int_{-a}^{a}\frac{f_k(t)dt}{t-z_j},\quad j=1,2;\quad k=1,2,3,4 \tag{3.3}$$

Substituting (3.3) into (1.10) and taking the limit as $y\to+0$, taking (3.2) and the following expression [10] into account

$$w_k(x,\pm0)=\pm\frac{1}{2}f_k(x)+\frac{1}{2\pi i}\int_{-a}^{a}\frac{f_k(t)dt}{t-x},\quad k=1,2,3,4$$

we obtain the relations

$$\sigma_{y1}(x,+0)=\sigma^0_{y1}(x)+f_1(x)/2+m_{11}S_2(x)+m_{21}S_4(x)$$
$$\sigma_{xy1}(x,+0)=\sigma^0_{xy1}(x)+f_2(x)/2-m_{31}S_1(x)+m_{41}S_3(x)$$
$$u_{,x}(x,+0)=u^0_{,x}(x)+f_3(x)/2-m_{51}S_2(x)+m_{61}S_4(x) \tag{3.4}$$
$$\upsilon_{,x}(x,+0)=\upsilon^0_{,x}(x)+f_4(x)/2-m_{71}S_1(x)+m_{81}S_3(x),\quad -\infty<x<\infty$$

Here

$$S_k(x)=\frac{1}{2\pi}\int_{-a}^{a}\frac{f_k(t)dt}{t-x},\quad k=1,2,3,4$$
$$m_{11}=m_{61}=\frac{p_1t_2-p_2t_1}{m},\quad m_{21}=\frac{t_2-t_1}{m},\quad m_{31}=m_{81}=t_1t_2m_{11},$$
$$m_{41}=t_1t_2m_{21},\quad m_{51}=\frac{p_1^2t_2-p_2^2t_1}{m},\quad m_{71}=t_1t_2m_{51},\quad m=t_1t_2(p_1-p_2)$$

The case when $\mu_1=\mu_2$. We have

$$(1+\kappa)\Phi_1(z)=w_1(z)-i\,w_2(z)+2Cw_3(z)+2iCw_4(z) \tag{3.5}$$
$$(1+\kappa)\Phi_2(z)=(\kappa-1)w_1(z)-i(1+\kappa)w_2(z)-4Cw_3(z)$$

Substituting (3.5) into (1.11) and taking the limit $y\to+0$, we obtain expressions which differ from (3.4) in that $m_{11}, m_{21}, \ldots, m_{81}$ are replaced by $m_{21}, m_{22}, \ldots, m_{82}$, where

$$m_{12}=m_{32}=m_{62}=\frac{\kappa-1}{2(1+\kappa)},\quad m_{22}=m_{42}=\frac{2C}{1+\kappa},\quad m_{52}=m_{72}=\frac{\kappa}{2C(1+\kappa)},\quad m_{82}=m_{11}$$

Substituting the relations obtained into the conditions of the interaction of a thin elastic inclusion with a piecewise-uniform medium (2.1) and taking (3.1) into account, we obtain the following system of four singular integro-differential equations

$$\lambda_{1k}S_2(\xi)+\lambda_{2k}S_4(\xi)+\Lambda_1\varphi_2(\xi)=F_1(\xi)$$
$$\lambda_{3k}S_2(\xi)+\lambda_{4k}S_4(\xi)+\Lambda_2\varphi_2(\xi)-\Lambda_3\varphi_4(\xi)=F_2(\xi)$$

$$\lambda_{5k}S_1(\xi)+\lambda_{6k}S_3(\xi)-\Lambda_3\varphi_3(\xi)=F_3(\xi) \tag{3.6}$$
$$\lambda_{7k}S_1(\xi)+\lambda_{8k}S_3(\xi)+\Lambda_3\varphi_1(\xi)=F_4(\xi),\quad |\xi|<1,\quad k=1,2$$

Here

$$S_j(\xi)=\frac{1}{2\pi}\int_{-1}^{1}\frac{\varphi_j'(\tau)d\tau}{\tau-\xi},\quad \varphi_j'(\xi)=f_j(a\xi),\quad j=1,2,3,4;\quad \xi=\frac{x}{a},\quad \tau=\frac{t}{a}$$

$$F_1(\xi)=-k_1\sigma_{y1}^0(a\xi)-u^{0'}(a\xi)+\Lambda_1 l_0 N(-a),\quad F_2(\xi)=-k_0\sigma_{y1}^0(a\xi)+\Lambda_2 l_0 N(-a)+\Lambda_3 V(-a)$$

$$F_3(\xi)=\sigma_{xy1}^0(a\xi)/\mu_0+\upsilon^{0'}(a\xi)+\Lambda_3 V(-a),\quad F_4(\xi)=-\sigma_{xy1}^0(a\xi)+T(-a)$$

$$\lambda_{1k}=k_1 m_{1k}-m_{5k},\quad \lambda_{2k}=k_1 m_{2k}+m_{6k},\quad \lambda_{3k}=\lambda_{4k}=k_0 m_{1k}$$

$$\lambda_{5k}=m_{7k}-m_{3k}/\mu_0,\quad \lambda_{6k}=m_{8k}-m_{4k}/\mu_0,\quad \lambda_{7k}=-m_{3k}$$

$$\lambda_{8k}=m_{4k},\quad k=1,2,\quad \Lambda_1=k_0/l_0,\quad \Lambda_2=k_1/l_0,\quad \Lambda_3=1/l_0,\quad l_0=l/a$$

The required functions $\varphi_j'(\xi)$ must satisfy the additional conditions

$$\int_{-1}^{1}\varphi_j'(\tau)d\tau=C_j,\quad j=1,2,3,4 \tag{3.7}$$
$$C_1=T(-a)-T(a),\quad C_2=N(-a)-N(a),\quad C_3=U(a)-U(-a),\quad C_4=V(a)-V(-a)$$

The axial force $N(\cdot)$, the cutting force $T(\cdot)$, the longitudinal displacement $U(\cdot)$ and the vertical displacement $V(\cdot)$ at the ends $w=\mp a$ of the inclusion are calculated from the a priori formulae [7].

In the case of an absolutely rigid inclusion ($\mu_0=\infty$) we have $f_3(x)=f_4'(x)=0$, and the system of equations (3.6) reduces to one singular integral equation

$$\frac{1}{2\pi i}\int_{-1}^{1}\frac{[\varphi_1'(\tau)+i\varphi_2'(\tau)]}{\tau-\xi}d\tau=-\frac{1}{m_{5k}}[u^{0'}(a\xi)-i\upsilon^{0'}(a\xi)],\quad |\xi|<1,\quad k=1,2 \tag{3.8}$$

If $\mu_0=0$, we obtain $f_1(x)=f_2(x)=0$, and the integral equation for the cut on the straight interface of the materials of the two media has the form

$$\frac{1}{2\pi i}\int_{-1}^{1}\frac{[\varphi_3'(\tau)+i\varphi_4'(\tau)]}{\tau-\xi}d\tau=\frac{1}{m_{2k}}[\sigma_{y1}^0(a\xi)-i\tau_{xy1}^0(a\xi)],\quad |\xi|<1,\quad k=1,2 \tag{3.9}$$

Integral equations (3.8) and (3.9) have the same structure and allow of a solution in closed form [10].

In the case of a thin-walled elastic inclusion we will represent the solution of (3.6) and (3.7) in the form

$$u_0'(\xi)=(1-\xi^2)^{-1/2}\sum_{n=0}^{\infty}X_n^jT_n(\xi),\quad |\xi|<1,\quad j=1,2,3,4 \tag{3.10}$$

($T_n(\cdot)$ are Chebyshev polynomials of the first kind). It then follows from (3.7) that $X_0^j=C_j/\pi$.

Substituting (3.10) into the integral equations (3.6), taking into account the fact that

$$\varphi_j(\xi)=X_0^j\left(\frac{\pi}{2}+\arcsin\xi\right)-(1-\xi^2)^{1/2}\sum_{n=1}^{\infty}\frac{X_n^j}{n}U_{n-1}(\xi),\quad j=1,2,3,4$$

($U_n(\cdot)$ are Chebyshev polynomials of the second kind), and using the usual procedure of the method of orthogonal polynomials, we obtain four infinite systems of linear algebraic equations, the first of which has the form (the summation is from $m=1$ to ∞)

$$\lambda_{1k}X_n^2+\lambda_2X_n^4-\Lambda_1\sum B_{n,m}X_m^2=\alpha_n^1-\Lambda_1 l_0X_0^2\beta_n$$
$$\alpha_n^j=\frac{4}{\pi}\int_{-1}^{1}F_j(\xi)U_{n-1}(\xi)\sqrt{1-\xi^2}d\xi,\quad j=1,2,3,4;\quad \beta_n=\pi\delta_{1,n} \tag{3.11}$$

$$B_{n,m} = \begin{cases} -\dfrac{8n[1+(-1)^{n+m}]}{\pi[m^2-(n-1)^2][n^2-(m+1)]^2}, & m \neq n-1, \quad m \neq n+1 \\ 0, & m = n-1, \quad m = n+1 \end{cases}$$

($\delta_{m,n}$ are Kronecker deltas). The remaining equations have a similar structure.

Using the estimates obtained previously [5] it can be shown that system (3.11) is quasi-regular for any values of the geometrical and physical parameters of the problem, and, consequently, we can use the reduction method on a computer to solve it. By determining the coefficients X^j_m (j = 1, 2, 3, 4) of expansions (3.10) we can use (3.3) and (3.5) to find the complex potentials $\Phi_j(\cdot)$ (j = 1, 2) and hence the stresses and displacements (1.10) and (1.11) at an arbitrary point of the composite.

Note that when $[\mu] = 0$, by putting in addition $\lambda_1 = \lambda_2$, we obtain from (3.6) the well-known results of the theory of thin elastic inclusions in a uniform medium [4, 5, 7].

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Translated by R.C.G.